results were also found to be independent of the region of the probe curve used as input. That is, input data taken at one volt increments from each of the regions indicated in Fig. 1 gave exact results. Over a reasonable voltage range the results were found to be independent of the number of data pairs input. Results for input data taken from regions 1 and 2 in Fig. 1 were exact when the voltage increment was increased to five volts which reduced the number of input data pairs from 26 to 6.

The sensitivity of the results to random error was investigated by superimposing a random error on the analytically generated input data. This investigation indicated that the results were sensitive to the region of the curve used as input as well as the voltage increment. Plasma property errors were found to be minimal when the input data were taken over a 20-25 v range (region 1 or 2) at a 1-volt increment.

Several Langmuir probe traces obtained from an operating ion thruster were analyzed by the graphical procedure[1] and the numerical procedure outlined above. In the analysis, the ion current to the probe was considered negligible and the probe current was assumed to be electron current only. Comparison of the plasma properties determined by these methods indicated good agreement for the plasma potential and Maxwellian number density, while poor agreement was observed for the Maxwellian electron temperature and primary electron energy and density. The average absolute difference, expressed as a percentage of the numerically determined plasma property, was 3%, 14%, 36%, 37%, and 285%, respectively. The numerical procedure determined electron temperatures and energies which are more consistent with expected values than those determined graphically. For example, on the thruster centerline the numerical procedure resulted in an average Maxwellian electron temperature and primary electron energy of 4.1 ev and 32 ev, respectively. The corresponding graphically determined averages were 2.4 ev and 16.2 ev. The Maxwellian electron temperature expected to exist in mercury bombardment thrusters is about 4-5 ev, while the primary electrons would be expected to have an energy in the 30-35 ev range when operating at a 37 volt anode potential.

Conclusions

Based on these studies, the following conclusions regarding the applicability of the numerical procedure are drawn: 1) For a given set of input data the results of the analysis are unique and independent of the initial estimate of electron temperature. 2) Convergence is fastest when the electron temperature is over-estimated. However, an estimate which is greater than twice the actual temperature may not result in convergence. 3) The results are sensitive to the region of the curve used as input and also to the voltage increment. Thus, in order to minimize the effect of random error the input data should be taken over a wide voltage range ($\sim 25V$) with a small enough voltage increment ($\sim 1V$) to give a fairly large number of input data pairs. 4) Plasma properties determined by the numerical procedure have either shown good agreement with those determined graphically or, where agreement was poor, have been closer to expected values.

With a suitable data acquisition system, the program could be used to provide real time plasma diagnostic information for an operating ion thruster. Although the procedure has been developed for thin plasma sheath analysis, it can easily be extended to thick sheath analysis since the nonlinear curve fitting technique is applied to the retarding field region of the probe trace which is independent of sheath dimensions.

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Turbulent Boundary Layers with Wall Injection

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RECENTLY, considerable attention has been given, from both the analytical and experimental points of view, to the development of a turbulent boundary layer with continuous injection or suction at the surface.[1-3] The more challenging situation of a turbulent boundary subjected to a step change in surface injection has been studied experimentally[4-7] and represents a rather severe test of the present calculation methods.

In this Note, the elegant method of Bradshaw, Ferriss, and Atwell[8] is used to calculate the perturbed flow region downstream of a discontinuity in surface injection. Although this method is not the most sophisticated now available, the solution of the differential equations for mean velocity and Reynolds shear stress (or, more accurately, turbulent kinetic energy) by the method of characteristics is well suited to the study of the propagation of the disturbance downstream of the surface change. Essentially, the turbulent energy equation is converted into an equation for shear stress by assuming three empirical functions relating the turbulent intensity, diffusion and dissipation to the shear stress profile. The relevance of these assumptions to the boundary layer with surface injection or suction is discussed in detail elsewhere.[9] Here it will suffice to state that κ, the von Karman constant, and A, the additive constant, in the logarithmic velocity profile

$$U/U_\tau = \kappa^{-1}\ln(yU_\tau/\nu + A$$

where U_τ is the friction velocity $= \tau_w^{1/2}$, τ_w is the kinematic wall shear stress, ν is the kinematic viscosity), were taken to be 0.40 and 2.0, respectively; i.e., the values generally accepted for smooth wall boundary layers with zero mass transfer and pressure gradient.

Before comparing the results of the method for the step change in surface injection, a comparison was made with the experimental data of Nayak[10] and of Wooldridge and Muzzy[11] for the turbulent boundary layers with uniform surface injection and zero pressure gradient. Satisfactory agreement between the calculated and experimental mean velocity profiles and Reynolds shear stress profiles was obtained. For the case of discontinuous surface injection, the data of Simpson[7] showed that the value of A could be used unchanged. Only small discontinuities were considered, in order not to invalidate the assumptions inherent in the calculation method of Bradshaw et al.[8] Comparisons were made with the experimental data of Levitch[6] and Simpson.[7] In the case of Levitch, the wall injection velocity V_w was decreased abruptly from 0.0045 U_1 (U_1 is the freestream velocity), upstream of the step to zero downstream. Good agreement was obtained for the mean velocity and shear stress

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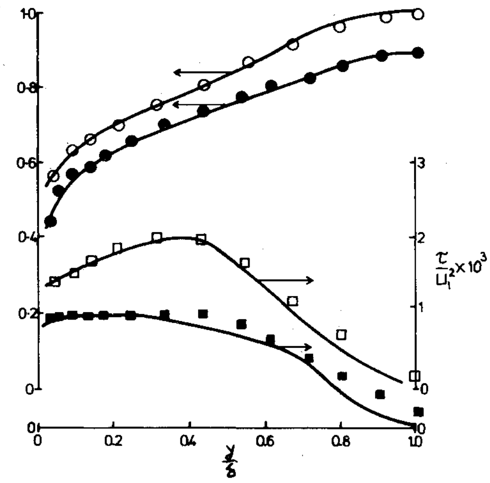

Fig. 1 Comparison between calculation and data of Levitch[6] for a discontinuation in surface injection. ○, U/U_l; □, τ/U_l^2; ____, calculation. Open symbols are for $X/\delta_0 = 13.8$, closed symbols are for $X/\delta_0 = 35.6$.

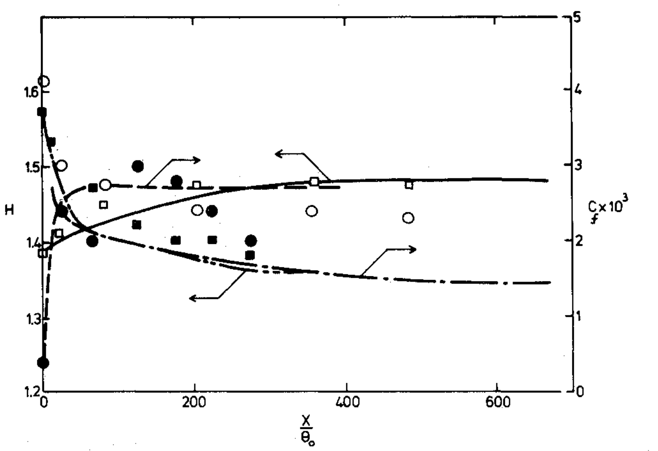

Fig. 2 Comparison between calculation of H and C_f with the data of Simpson[7] (open symbols) and McQuaid[5] (closed symbols). ○, C_f; □, H. Simpson[7] ($V_w = 0$ upstream, $V_w = 0.002\ U_l$ downstream): calculation ____, H; _ _ _ _ C_f. McQuaid[5] ($V_w = 0.00\,34\ U_l$, upstream, $V_w = 0$ downstream): calculation _ _ _ _, H; _ _ _ _, C_f. Note that θ_0 is the momentum thickness of the layer at $x = 0$.

profiles (Fig. 1). However, although the calculated and measured shape parameter H ($\equiv \delta^*/\theta$, where δ^*, θ are the displacement and momentum thicknesses, respectively, of the layer) values were in good agreement, the calculated skin friction coefficient C_f ($\equiv 2\tau_w/U_l^2$) overestimated the measured values by as much as 40% for $X/\delta_0 > 25$. (δ_0 is the thickness of the layer immediately upstream of the discontinuity and X is the streamwise distance measured from the discontinuity). The fact that the shear stress profile at $X/\delta_0 = 35.6$ (Fig. 1) bears little resemblance to that in the self-preserving layer, suggests that the boundary layer recovers rather slowly to the sudden decrease in wall blowing. Antonia and Luxton[12] noted a similar effect for a sudden decrease in wall roughness. For Simpson's case there was a sudden increase in wall injection from $V_w = 0$ upstream of the step to $V_w = 0.002\ U_l$ downstream. Good agreement (Fig. 2) was again obtained for the shape parameter but the calculated C_f now underestimated the measured values by as much as 34%.

The hyperbolic nature of the equations used by Bradshaw et al.[8] suggests that the position of the internal layer coincides with the outgoing characteristic from the position at which the wall injection changes. Denoting the height of the internal layer by δ_i, the equation for the outgoing characteristic (when the advection and diffusion terms are neglected from the turbulent energy equation) reduces to

$$d\delta_i/dx = (2a_l\tau_w)^{1/2}/U$$

where U, τ_w refer to the mean velocity profile and wall shear stress of the layer immediately upstream of the step and a_l is the ratio $\tau/\bar{q}^2$ ($\bar{q}^2$ is the turbulent energy) assumed unaffected by the blowing rate. The rate of growth of the internal layer is thus entirely determined by upstream conditions, a result which seems to be supported by the experimental data of Simpson.[7] With the internal layer confined within the logarithmic region of the upstream boundary layer,

$$U/U_\tau = \kappa^{-1}\ln(y/z_0)$$

where $z_0 (= \nu/U_\tau e^A)$ can be interpreted as the characteristic roughness scale of the layer. An implicit expression for the variation of δ_i with X can be written as

$$\delta_i(\ln(\delta_i/z_0) - 1) = \kappa(2a_l)^{1/2}X$$

In the case of Simpson's data for the unblown step change in surface condition, the preceding relation, with a_l, assumed constant equal to 0.15, yields a value of $36\delta_0$ as the distance from step to the position where $\delta_\theta \simeq \delta_0$. Although this estimate can only be taken as a rough approximation in view of the assumed logarithmic profile, it compares well with the value of $37\delta_0$ estimated by Simpson[7] who integrated the equation from the outgoing characteristic from the step using the mean velocity profile and included the contribution from the advection and diffusion terms. For the case of Levitch[6] Simpson estimates the relaxation length to be about $20\delta_0$, in contrast with the results of Fig. 1 which clearly show that, even at $35\delta_0$, the measured Reynolds shear stress has still not recovered from the change in surface condition. It is not unreasonable to expect the Reynolds shear stress distribution to recover more slowly to the change in surface condition than the mean velocity profile and any attempt to estimate the relaxation length of the layer should probably aim at estimating the relaxation distance for the turbulence structure.

McQuaid[4] measured mean velocity profiles downstream of a discontinuation in wall injection ($V_w/U_l = 0.0034$ upstream of the step), and concluded that the fully developed state is not quite reached at the last measuring station $33\delta_0$ downstream from the discontinuity. The conclusion was reached on the basis that the experimental values of C_f, H, and R_θ ($\equiv U_l\theta/\nu$) did not conform to the zero injection, zero pressure gradient results. It supports the conclusion drawn from Levitch's results that a relaxation distance well in excess of $35\delta_0$ may be required for the recovery of the layer to a step down in wall injection. It is clearly in contrast with Tani's[13] estimate of $16\delta_0$ for this relaxation distance. Lili and Michel[14] and Chen[15] have calculated mean velocity profiles in good agreement with the experimental data of McQuaid[5] The method of Bradshaw et al. has been used to calculate the velocity and Reynolds stress fields for McQuaid's experiment. The calculation was started at the position of discontinuation of injection, with the measured mean velocity profile but with a Reynolds stress profile calculated from the momentum equation, using the velocity field upstream of the step. The calculated values of C_f and H are shown in Fig. 2. The skin friction is in reasonable agreement with experimental data, obtained by the momentum integral equation (McQuaid has stated that the experimental uncertainty in this estimate of C_f may be due to the combination of a residual injection rate downstream of the step and of a locally nonzero pressure gradient). The calculated values of H are slightly below the experimental results, a result also obtained by Chan[15] who used an implicit finite difference scheme to solve the boundary-layer equations with a mixing length formulation derived from the turbulent kinetic energy equation.

Although the experimental determination of C_f in the region near the step is suspect, the discrepancy between the calculated and measured C_f well downstream of the step need further investigation. Available experimental results appear to suggest that the relaxation length downstream of either a step up or step down in wall injection is well in excess of $35\delta_0$ when the relaxation of the turbulence structure is also taken into account.

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Transient Conduction in a Finite Slab with Variable Thermal Conductivity

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Nomenclature

A_o $= B_I\delta/(2+B_I\delta)$
b = slab thickness
B_I = Biot number $= hb/k_o$, nondimensional
h = heat transfer coefficient
C_p = specific heat
$k(\theta)$ $= k_o(1+\beta\theta)$
k_o = reference thermal conductivity at $T=T_o$
$K(\theta)$ $= k(\theta)/k_o$
t = nondimensional time $= \alpha_o\tau/b^2$
t_δ = value of t at $\delta=1$ for linear case
T = temperature
T_g = driving gas temperature
x = space coordinate
z $= \eta/[2\sqrt{t},]$ dimensionless
α_o = reference thermal diffusivity, $k_o/(\rho C_p)$
β = constant (thermal conductivity coefficient)
δ = penetration depth
λ = constant adjustable parameter
ρ = density
τ = time
η $= b/x$, nondimensional
ψ = initial constant property solution
Θ = nondimensional temperature $= (T-T_o)/(T_g-T_o)$

Subscripts

$(\ )_x$ = partial derivative with respect to x
$(\ )_t$ = partial derivative with respect to t
$(\ )_\eta$ = partial derivative with respect to η

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I. Introduction

THERMAL design of heat sink rocket nozzles requires the transient temperature distribution in the finite wall subjected to Newtonian heating at the exposed surface and negligible heat loss from the outer. Because of the high temperature range involved and considerable variation of thermal conductivity with temperature for the commonly employed materials, the customary assumption of constant average property can lead to significant inaccuracies in the desired wall thickness. It is therefore imperative that the thermal conductivity variation is invariably taken into account in the analysis.

Unfortunately no exact solution is possible for this problem, owing to the nonlinearity of the differential equation. To the authors' knowledge, even an approximate solution has not yet been reported in the literature. Moreover, the determination of thermal conductivity over wide temperature range of the present-day materials employed in the space program is relatively difficult and exhaustive by the conventional methods, but is best accomplished by the method of "nonlinear estimation"[1] which is gaining an increasing role in engineering design. Numerical method, if employed for this parameter estimation program, will demand considerable computer time and may not always justify its use.

It is the purpose of this Note to develop therefore a simple closed-form approximate solution that will be useful for the thermal design of rocket nozzles following the nonlinear parameter estimation. Linear variation of thermal conductivity with temperature is assumed here.

II. Analysis

A. Semi-Infinite Domain ($t\le t_\delta$)

The equations to be solved are

$$\Theta_t = (K(\Theta)\Theta_\eta)_\eta\,;\ 0\le\eta\le 1,\ t>0 \tag{1}$$

$$\theta_\eta(o,t) = B_I[\theta(o,t)-1] \tag{2}$$

$$\theta(\delta,t) = 0 \tag{3}$$

$$\theta(\eta,o) = 0 \tag{4}$$

We now seek to find the solution of Eqs. (1-4) by the method of "optimal linearization," recently applied by Vujanovic[2] for heat transfer problems. The initial solutions $\theta=\psi$ for the constant conductivity case are obtained from Ref. 3, using integral methods with parabolic profile assumed